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The One-Site Distribution of Gibbs States on Bethe Lattice Are Probability Vectors of Period $\leqslant 2$ for a Nonlinear Transformation

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We prove that the one-site distribution of Gibbs states (for any finite spin set S) on the Bethe lattice is given by the points satisfying the equation $\pi = T^2\pi$, where $T = h \circ A \circ \varphi$, with $\varphi(x) = x^{(q-1)/q}$, $h(x) = (x/\|x\|_q)^q$, $A = (a(r, s): r, s \in S)$, and

$$a(r, s) = \exp(K[r, s] + (1/q)[N, r + s])$$

We also show that for A a symmetric, irreducible operator the nonlinear evolution on probability vectors $x(n+1) = Ax(n)^p/\|Ax(n)^p\|_1$ with $p > 0$ has limit points ξ of period $\leqslant 2$. We show that A positive definite implies limit points are fixed points that satisfy the equation $A\xi^p = \lambda\xi$. The main tool is the construction of a Liapunov functional by means of convex analysis techniques.

KEY WORDS: Gibbs states; Bethe lattice; spin vector; dynamical systems; automata networks; subdifferential; cyclically monotone function; convex function; Liapunov functional.

1. INTRODUCTION

We study nonlinear dynamics having associated limit points of period $t \leqslant 2$. The main applications of these results are made in the description of the Gibbs states of the Bethe lattice. We also show that results obtained can be viewed as a nonlinear generalization of symmetrical Markov chains.

In Section 2 we introduce the mathematical techniques we use. They were first developed[1,3] to analyze the dynamical behavior of automata networks related to neuronal activity models. The dynamical systems we study can be written as $T = h \circ A \circ \varphi$, where A is a symmetric operator and

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$f = \varphi \circ h$ is in the subdifferential of a convex function. The fundamental tools to analyze these systems are the Liapunov functionals, which are constructed in Theorem 1. To obtain them we use concepts from convex analysis. In Theorem 2 we discuss conditions that, imposed on the Liapunov functionals, imply that finite orbits can only have period $\leqslant 2$. We can also establish hypotheses that allow us to assert that finite orbits are only fixed points and some other hypotheses implying that there is only one fixed point, any other finite orbit being of period 2 (Theorem 3). We finish Section 2 by characterizing the relevant functions in our applications, those that belong to subdifferentials of seminorms. In this case the Liapunov functional takes a simple form (Theorem 4). Some of these results were established in a more restricted framework in refs. 1–3.

In Section 3 we study the Gibbs states ν on the configuration space $S^{L_\infty(q)}$, where S is a finite spin set and $L_\infty(q)$ the Bethe lattice of coordination number q. We assume that ν is obtained by the thermodynamic limit taken with $L_n(q) \to L_\infty(q)$, with $L_n(q)$ the Cayley tree of coordination number q constructed until the shell n. The Hamiltonian (for $S^{L_n(q)}$) is

$$H_n(q) = -\left(K \sum_{i,j} [\sigma(i), \sigma(j)] + \sum_i [N, \sigma(i)]\right)$$

where $\sum_{i,j}$ is over neighbor sites, K is the interaction, and N is the exterior magnetic field. We prove that the probability vector $\pi = (\nu\{\sigma: \sigma(0) = r\}, r \in S)$ satisfies the equation $\pi = (h \circ A \circ \varphi)^2 \pi$, where $\varphi(x) = x^{(q-1)/q}$, $h(x) = (x/\|x\|_q)^q$, and A is the matrix given by

$$a(r, s) = \exp(K[r, s] + (1/q)[N, r+s]) \qquad \text{for} \quad r, s \in S$$

(Theorem 5). In obtaining this equation we use the results of the present section and the recursive equations for the Gibbs states on $S^{L_n(q)}$ obtained in ref. 7. By using the same method as in Theorem 3, we deduce that for large ferromagnetic iteration $K \gg 0$ (it suffices that $K > 0$ for $|S| = 2$) and under the absence of exterior magnetic field, $N = 0$, the equation for π is reduced to $\pi = (h \circ A \circ \varphi)\pi$.

In Section 4 we study some nonlinear dynamics on $\mathbb{R}_+^s$ whose limit points are orbit of period $\leqslant 2$. This allows us to assert in Corollary 3 that the nonlinear evolution

$$\pi_r(n+1) = \frac{\sum a(r, s)\pi_r(n)^p}{\sum_{r' \in S} \sum_{s \in S} a(r', s)\pi_{r'}(n)^p} \qquad n \geqslant 0, \quad r \in S$$

of probability vectors has only limit points of period $t \leqslant 2$, where $(a(r, s): r, s \in S)$ is symmetric. This generalizes the equations obtained for limit points of symmetric Markov chains, which in fact satisfy the equation $\pi = A^2\pi$; recall that A is symmetric, so its period is $\leqslant 2$. (But the results known for Markov chains are stronger because they characterize completely the domain of attraction of any one of the probability vectors satisfying $\pi = A^2\pi$ and also they describe these points.) In Corollary 4 we obtain similar results for other transformations than $\varphi(x) = x^p$ which also give rise to period $\leqslant 2$ behavior of limit probability points (we use Orlicz space theory).

In Section 5 we establish the consequence of our results when we study evolution equations Ax^p in $\mathbb{R}^S$. It induces a transformation on the rays that also have period $\leqslant 2$ limit behavior when A is symmetric. We also give a bound for the number of solutions of the one-site distribution of Gibbs states when $|S| = 2$ and we detail these solutions for case $q = 2$ $[L_\infty(2) = \mathbb{Z}]$ and $S = \{0, 1\}$. For a deeper description of solutions in the case $S = \{0, 1\}$, $q \geqslant 2$, see ref. 6, and ref. 9 for the case $q = 2$. Results in the case $|S| \geqslant 2, q = 3$ were first obtained in ref. 8. Recent relevant results, obtained in a more general framework can be found in ref. 10.

2. PRELIMINAR RESULTS

The evolution equation we shall study takes the following form in a set D included in a real Hilbert space $(H, \langle\ \rangle)$:

$$x(n+1) = Tx(n), \qquad T = h \circ A \circ \varphi, \qquad \text{for} \quad x(0) \in D \tag{1}$$

where $\varphi\colon D \subset H \to F \subset H$, $A\colon H \to H$, and $h\colon F \to D$ satisfy

$$A \text{ is a symmetric linear operator such that } A(F) \subset F \tag{2}$$

$$f = \varphi \circ h \quad \text{is cyclically monotone on the closed convex set } F \tag{3}$$

which means that for any cycle $(u_0,..., u_{m-1}, u_m = u_0) \in F^{m+1}$, $m \geqslant 2$, we have

$$\sum_{i=0}^{m-1} \langle f(u_{i+1}) - f(u_i), u_{i+1} \rangle \geqslant 0 \tag{4}$$

We call f strictly cyclically monotone if it is cyclically monotone and for any nontrivial cycle $(u_0,..., u_{m-1}, u_m = u_0)$ such that there exists some $0 \leqslant k \leqslant m-1$ with $u_k \neq u_{k+1}$, we have

$$\sum_{i=0}^{m-1} \langle f(u_{i+1}) - f(u_i), u_{i+1} \rangle > 0 \tag{5}$$

It can be shown that f is cyclically monotone iff it belongs to the subdifferential of a convex potential, so there exists $g: F \to \mathbb{R}$ convex such that

$$g(u) \geqslant g(v) + \langle f(v), u - v \rangle, \qquad \forall u, v \in F \tag{6}$$

f is strictly cyclically monotone iff g is strictly convex. In this case we have[2]

$$g(u) - g(v) = \langle f(v), u - v \rangle \qquad \text{implies} \quad u = v \tag{7}$$

Recall $g^*: H \to \mathbb{R} \cup \{-\infty, \infty\}$, the polar of g, satisfies

$$g^*(v) = \sup_{u \in F} (\langle u, v \rangle - g(u)) \tag{8}$$

Then, since g is a convex potential of f, we deduce

$$g^*(f(v)) + g(v) = \langle v, f(v) \rangle \qquad \text{for} \quad v \in F \tag{9}$$

The main results characterizing the finite orbits of the dynamical system $T = h \circ A \circ \varphi$ are as follows.

Theorem 1. Let A, φ, and h satisfy (2) and (3); then

$$H(x(n)) = -g^*(\varphi(x(n))) + g(A\varphi(x(n))) \nearrow \text{ with } n \geqslant 1 \tag{10}$$

so $H(x(n))$ is a Liapunov functional increasing with $n \geqslant 1$.

Theorem 2. Let A, φ, and h satisfy (2), (3), and (5), so $\varphi \circ h$ is strictly cyclically monotone. Then, if $(x(0),..., x(t-1), x(t) = x(0))$ is a periodic orbit, then it is of length $t \leqslant 2$.

Theorem 3. Let A, φ, and h satisfy (2) and (3) such that any periodic orbit is of length $t \leqslant 2$ [for instance, if $f = \varphi \circ h$ satisfies (5)]; then: (a) if A is positive definite, then the periodic orbits of T are only fixed points ($t = 1$); (b) if A is negative definite, $0 \in F$, and $\varphi \circ h(0) = 0$, then $h(0)$ is the only fixed point of T and any other periodic orbit is of length $t = 2$.

Proof of Theorem 1. From (9)

$$\begin{aligned}
-g^*(\varphi(x(n+1))) &= -g^*(\varphi \circ h(A\varphi(x(n)))) \\
&= g(A\varphi(x(n))) \\
&\quad - \langle A\varphi(x(n)), \varphi(h(A\varphi(x(n)))) \rangle \\
&= g(A\varphi(x(n))) \\
&\quad - \langle A\varphi(x(n)), \varphi(x(n+1)) \rangle
\end{aligned}$$

Then

$$H(x(n+1)) = -\langle A\varphi(x(n)), \varphi(x(n+1))\rangle + g(A\varphi(x(n))) + g(A\varphi(x(n+1))) \quad (11)$$

For $n \geqslant 1$ define

$$\Delta(n) = H(x(n+1)) - H(x(n)) \quad (12)$$

If we put $u(n) = A\varphi(x(n))$, we get

$$\varphi(x(n+1)) = \varphi \circ h(A\varphi(x(n))) = f(u(n))$$

By symmetry of A we obtain

$$\Delta(n) = -\langle f(u(n)), u(n+2) - u(n)\rangle + g(u(n+2)) - g(u(n)) \quad (13)$$

Since g is a potential of f, we deduce $\Delta(n) \geqslant 0$, $\forall n \geqslant 1$; then, $H(x(n))$ increases for $n \geqslant 1$. ∎

Proof of Theorem 2. If $(x(0),..., x(t-1), x(t) = x(0))$ is a periodic orbit and the functional $H(x(n))$ is constant for $n \geqslant 0$, then $\Delta(n) = 0$. From (13) and the strictness condition on f we deduce $u(n+2) = u(n)$. By applying h, we get $x(n+3) = x(n+1)$ for any $n \geqslant 0$, then the result. ∎

Proof of Theorem 3. Let $(x(0), x(1))$ be the periodic orbit; then $x(n) = x(n (\bmod 2))$.

(a) Let

$$\begin{aligned}\gamma &= \langle \varphi(x(0)) - \varphi(x(1)), A(\varphi(x(0)) - \varphi(x(1)))\rangle \\ &= \langle \varphi(x(0)), A\varphi(x(0)) - A\varphi(x(1))\rangle \\ &\quad + \langle \varphi(x(1)), A\varphi(x(1)) - A\varphi(x(0))\rangle\end{aligned}$$

Now since we have a 2-periodic orbit $x(n+1) = x(n-1)$, we deduce

$$\begin{aligned}&\langle \varphi(x(n)), A\varphi(x(n)) - A\varphi(x(n+1))\rangle \\ &\quad = \langle \varphi \circ h(A\varphi(x(n+1))), A\varphi(x(n)) - A\varphi(x(n+1))\rangle \\ &\quad \leqslant g(A\varphi(x(n))) - g(A\varphi(x(n+1)))\end{aligned}$$

Then $\gamma \leqslant 0$. since A is positive definite, we deduce $\varphi(x(0)) = \varphi(x(1))$; by applying $h \circ A$, we obtain $x(1) = x(0)$.

(b) From the hypothesis, $h(0)$ is a fixed point. Now, suppose $x(1)=x(0)$ is a fixed point. We have

$$\begin{aligned} g(0) &\geqslant g(A\varphi(x(0))) + \langle \varphi \circ h(A\varphi(x(0))), -A\varphi(x(0)) \rangle \\ &= g(A\varphi(x(0))) - \langle \varphi(x(0)), A\varphi(x(0)) \rangle \end{aligned}$$

Since $\varphi \circ h(0) = 0$ and g is a convex potential of $\varphi \circ h$ from (9), we conclude that $g(0)$ is a global minimum of g, so $\langle \varphi(x(0)), A\varphi(x(0)) \rangle \geqslant 0$. Since A is negative definite we get $\langle \varphi(x(0)), A\varphi(x(0)) \rangle = 0$; then $\varphi(x(0)) = 0$. By applying $h \circ A$, we obtain $x(0) = h(0)$. ▌

The most important applications concern functions φ, h such that $f = \varphi \circ h$ belongs to the subdifferential of a seminorm g (as occurs for some cellular automata where f is the sign function[2,3]). In this case the strictness hypothesis of Theorem 2 is not satisfied, but for some of these functions we will be able to show the conclusions of Theorem 2, then also those of Theorem 3 (we do this in the following sections). Now let us describe this class of subdifferentials and characterize the Liapunov functional $H(x(n))$ for a class of functions containing them.

Theorem 4. If $f = \varphi \circ h$ satisfies

$$\langle f(u) - f(v), u \rangle \geqslant 0 \qquad \forall u, v \in F \tag{14}$$

then

$$g(u) = \langle u, f(u) \rangle, \qquad u \in F \tag{15}$$

is a convex potential of f and

$$H(x(n)) = g(A\varphi(x(n))) \nearrow \text{ for } n \geqslant 1 \tag{16}$$

Furthermore, if f satisfies (14) for $F = H$ and also satisfies

$$\langle f(u), u \rangle = -\langle f(-u), u \rangle \qquad \forall u \in H \tag{17}$$

then its potential $g(u) = \langle u, f(u) \rangle$ is a seminorm. Conversely, if f belongs to the subdifferential of a seminorm g, then $g(u) = \langle u, f(u) \rangle$ and f satisfies (17) and (14) for $F = H$.

Proof. From (14) we obtain directly $g(v) \geqslant g(u) + \langle f(u), v - u \rangle$; then g is a potential of f. From (9) and (15) we get $g^*(f(v)) = 0$. Since

$$\varphi(x(n)) = \varphi \circ h(A\varphi(x(n-1))) = f(u(n-1))$$

for $n \geqslant 1$ we get $g^*(\varphi(x(n))) = 0$. By replacing it in (10), we get (16).

Now, if f satisfies (14) and (17) it is easy to show that g is a seminorm.[2] Conversely, if $g(u)$ is a seminorm, then it is convex and any of its subdifferentials f satisfy

$$g(2u) = 2g(u) \geqslant g(u) + \langle u, f(u)\rangle, \quad g(0) = 0 \geqslant g(u) - \langle u, f(u)\rangle$$

Then $g(u) = \langle u, f(u)\rangle$. Equality (17) follows from $g(u) = g(-u)$, and inequality

$$\langle v, f(v)\rangle = g(v) \geqslant g(u) + \langle f(u), v-u\rangle = \langle u, f(u)\rangle + \langle f(u), v-u\rangle$$

implies (14) for $F = H$. ▌

3. APPLICATIONS IN THE BETHE LATTICE

Let $q \geqslant 2$. Consider the graph constructed as follows: we start from a central point 0 and we add q points all connected to 0. Call these new points the shell 1. The shell $k+1$ is constructed by connecting $q-1$ new points to any point of the shell k. The graph constructed until the shell n is denoted $L_n(q)$ and the infinite graph obtained for $n \to \infty$, denoted $L_\infty(q)$, is called the Cayley tree of coordination number q. Recall that $L_\infty(2) = \mathbb{Z}$.

For $L_n(q)$ the number of points of the boundary and the number of points of the graph grow like $(q-1)^n$. Then some problems appear when we try to describe the thermodynamic limits of Ising models on $L_\infty(q)$. We shall only consider those thermodynamic limits arising from limits $L_{n'}(q) \to L_\infty(q)$, $n' \to \infty$ [and not from other sequences $\Lambda_n \to L_\infty(q)$, $n \to \infty$].[5]

When we consider only local properties infinitely far from the boundary in the limit $n \to \infty$, as we shall do, all sites of $L_\infty(q)$ are equivalent and it is called the Bethe lattice.

Let $S \subset \mathbb{R}^d$ be a finite spin set; we denote by $[\ ,\ ]$ the inner product on $\mathbb{R}^d$. We shall consider $\Omega_n = S^{L_n(q)}$ for $n \in \mathbb{N} \cup \{\infty\}$ (Ω_∞ is the configuration space) and by $\alpha_{n,m}: \Omega_m \to \Omega_n$, $\sigma \to \sigma|_{\Omega_n}$ the restriction of σ to the coordinates of Ω_n for $n \leqslant m \leqslant \infty$.

For n finite the partition function on $L_n(q)$ is given by

$$Z_n = \sum_{\sigma \in \Omega_n} \exp[-H_n(\sigma)], \quad H_n(\sigma) = -\left\{K \sum_{(i,j)} [\sigma(i), \sigma(j)] + \sum_i [N, \sigma(i)]\right\} \tag{18}$$

where the sum $\sum_{(i,j)}$ is over all neighbor couples (i, j) and $\sum_i$ over all the sites i of the tree $L_n(q)$. The parameter $K \in \mathbb{R}$ is the interaction ($K > 0$ if it is ferromagnetic, $K<0$ if it is antiferromagnetic) and $N \in \mathbb{R}^d$ is the exterior magnetic field.

The Gibbs law on Ω_n is denoted by μ_n; it satisfies

$$\mu_n\{\sigma \in \Omega_n : \sigma(i) = r_i,\, i \in L_n(q)\}$$
$$= Z_n^{-1} \sum_{\{\sigma \in \Omega_n : \sigma(i) = r_i,\, i \in L_n(q)\}} \exp[-H_n(\sigma)]$$

As $L_n(q) \to L_\infty(q)$, $n \to \infty$, by diagonal arguments there exists a subsequence $n' \to \infty$ such that the following limits exist (in the sense of vague convergence of measures):

$$\lim_{n' \to \infty} \alpha_{n,n'} \mu_{n'} = \nu_n, \qquad \forall n \geqslant 1 \tag{19}$$

Furthermore, there exists a unique measure ν on the configuration space Ω_∞ such that

$$\alpha_{n,\infty} \nu = \nu_n \tag{20}$$

The measure ν is a Gibbs state.[5]

We shall study the one-site distribution of this class of Gibbs states:

$$\pi = (\pi_r : r \in S), \qquad \pi_r = \nu\{\sigma \in \Omega_\infty : \sigma(0) = r\} \tag{21}$$

Since all sites in the Bethe lattice are equivalent we have

$$\pi_r = \nu\{\sigma \in \Omega_\infty : \sigma(i) = r\} \qquad \text{for any} \quad i \in L_\infty(q)$$

Theorem 5. Let ν be a Gibbs state on $\Omega_\infty = S^{L_\infty(q)}$ given by (19), (20). Then the one-site distribution of this Gibbs state which is given by (21) is a probability vector π, which satisfies the equation

$$\pi = (h \circ A \circ \varphi)^2 \pi \tag{22}$$

where h, A, and φ act on $\mathbb{R}_+^s \backslash \{0\}$ and are given by

$$A = (a(r, s) : r, s \in S), \qquad a(r, s) = \exp\left(K[r, s] + \frac{1}{q}[N, r+s]\right) \tag{23}$$

$$\varphi(x) = x^{(q-1)/q}, \qquad h(x) = \left(\frac{x}{\|x\|_q}\right)^q \tag{24}$$

where $x^p = (x_r^p : r \in S)$, and $\|\cdot\|_q$ is the L^q-norm.

Proof. Let ν be a Gibbs state and (ν_n), $n' \to \infty$, satisfying (19), (20). We note that $A_n(r) = \{\sigma \in \Omega_n : \sigma(0) = r\}$ for $n \in \mathbb{N} \cup \{\infty\}$. We have

$$\nu_n(A_n(r)) = \nu(\alpha_{n,\infty}^{-1}(A_n(r))) = \nu(A_\infty(r)) = \pi_r$$

On the other hand,

$$\nu_n(A_n(r)) = \lim_{n' \to \infty} (\alpha_{n,n'}\mu_{n'})(A_n(r)) = \lim_{n' \to \infty} \mu_{n'}(A_{n'}(r))$$

Note that $\pi(n) = (\pi_r(n): r \in S)$, where $\pi_r(n) = \mu_n(A_n(r))$; then

$$\pi = \lim_{n' \to \infty} \pi(n') \tag{25}$$

Let $\Gamma = \{\pi(n): n \in \mathbb{N}\}$, so if we also show that the set of accumulation points of Γ satisfy (22), the theorem will be proved.

By the construction of the Cayley tree it can be shown[7] that there exist functions $W_n(r)$ such that the following relations hold:

$$\pi_r(n) = \exp[N, r](W_r(n))^q \left\{ \sum_{s \in S} \exp[N, s](W_s(n))^q \right\}^{-1} \tag{26}$$

$$W_r(n) = \sum_{s \in S} \exp(K[r, s] + [N, s])(W_s(n-1))^{q-1} \tag{27}$$

Let us denote

$$c_n = \sum_{s \in S} \exp[N, s](W_s(n))^q$$

Then we get

$$\pi_r(n)^{1/q} = \sum_{s \in S} \exp(K[r, s] + [N, s]) \exp\left(\frac{1}{q}[N, r]\right)$$
$$\times c_n^{-1/q} \pi_s(n-1)^{(q-1)/q} \exp\left(-\frac{q-1}{q}[N, s]\right) c_{n-1}^{(q-1)/q}$$

Then

$$\pi_r(n)^{1/q} = (c_n^{-1} c_{n-1}^{q-1})^{1/q} \sum_{s \in S} \exp(K[r, s])$$
$$\times \exp\left(\frac{1}{q}[N, r+s]\right) \pi_s(n-1)^{(q-1)/q}$$

From $\sum_{r \in S} \pi_r(n) = 1$ we have $\|\pi(n)^{1/q}\|_q = 1$; then necessarily

$$(c_n^{-1} c_{n-1}^{q-1})^{1/q} = \|A(\pi(n-1))^{(q-1)/q}\|_q^{-1}$$

(remark that A maps $\mathbb{R}_+^S \backslash \{0\}$ into itself). By using (23) and (24) we finally get

$$\pi(n) = (h \circ A \circ \varphi)\pi(n-1) \tag{28}$$

Take $T = h \circ A \circ \varphi$; it maps $\mathbb{R}_+^S \backslash \{0\}$ in the simplex $P_1 = \{y \in \mathbb{R}_+^S : \sum_{r \in S} y_r = 1\}$; then its limit set Θ_1 is given by

$$\begin{aligned}\Theta_1 = \{ y \in P_1 : \exists x \in P_1, &\text{ a subsequence } n'' \to \infty \\ &\text{such that } y = \lim_{n'' \to \infty} T^{n''} x \}\end{aligned} \tag{29}$$

We have that the set of accumulation points of Γ is included in Θ_1; then the theorem will follow when we show that any point $y \in P_1$ satisfies Eq. (22), that is, $y = T^2 y$. This will constitute our next result.

Now let us extend φ, h to $F = \mathbb{R}_+^S$ such that $h(0) = \varphi(0) = 0$; then

$$f(x) = \varphi \circ h(x) = \left(\frac{x}{\|x\|_q}\right)^{q-1} \qquad \text{if} \quad x \neq 0, \quad f(0) = 0 \tag{30}$$

belongs to the subdifferential of $g(x) = \|x\|_q$. The limit set of T acting on $\mathbb{R}_+^S$ is $\{0\} \cup \Theta_1$.

Take $y \in \Theta_1$; then $\exists x \in P_1$ and $n'' \to \infty$ such that $y = \lim_{n'' \to \infty} T^{n''} x$. We have that T maps the compact P_1 into itself; then there exists a subsequence $n''' \subset (n''-1)$ such that there exists $z = \lim_{n''' \to \infty} T^{n'''} x$. By continuity of T on P_1 we get $Tz = y$.

Note $v(n) = A\varphi(T^n z)$. The operator A is symmetric; then we can apply (16), so $\|A\varphi(T^n x)\|_q$ increases with $n \geqslant 1$ to a finite quantity M. By continuity, $\|v(n)\|_q = M$, $\forall n \geqslant 0$. And also by continuity on $\Delta(n)$ [see (12), (13)] we get $\langle f(v(2)), v(2) - v(0) \rangle = 0$. Then we obtain

$$\|v(2)\|_q = \|v(0)\|_q + \langle f(v(0)), v(2) - v(0) \rangle \tag{31}$$

Now, from the equalities

$$\langle f(v(0)), v(0) \rangle = \|v(0)\|_q, \qquad \|v(0)\|_q^{q-1} = \|v(0)^q\|_{q/(q-1)}$$

expression (31) becomes

$$\|v(2)\|_q \, \|v(0)^{q-1}\|_{q/(q-1)} = \langle v(0)^{q-1}, v(2) \rangle \tag{32}$$

that is, the Holder inequality becomes an equality; then there exist λ, $\lambda' \neq 0$ such that $\lambda(v(2))^q = \lambda'(v(0)^{q-1})^{q/(q-1)}$, which implies $v(2) = \lambda'' v(0)$. Since $\|v(2)\|_q = \|v(0)\|_q$, we deduce $v(2) = v(0)$; then $A\varphi(z(2)) = A\varphi(z(0))$. By applying h, we deduce $z(3) = z(1)$, that is, $T^2 y = y$. This finishes the proof. ▌

Remark. If we make the change of variables $\xi(n) = \pi(n)^{1/q}$, then $\xi(n)$ satisfies the evolution equation

$$\xi(n+1) = \frac{A(\xi(n)^{q-1})}{\|A(\xi(n)^{q-1})\|_q}$$

In this case $\varphi(x) = x^{q-1}$, $h(x) = x/\|x\|_q$ if $x \neq 0$, $h(0) = 0$. Also, $\|\xi(n)\|_q = 1$ if $\xi(0) \neq 0$. Then $\varphi \circ h = (x/\|x\|_q)^{q-1}$, which is the same as (30), that is, $\varphi \circ h$ is invariant under change of variables, as it should be. Then we can prove directly that the limit points ξ of $(\xi(n): n \geqslant 0)$ satisfy $\xi = (h \circ A \circ \varphi)^2 \xi$, $\|\xi\|_q = 1$ (for solutions $\xi \neq 0$).

For large ferromagnetic interaction and in the absence of exterior magnetic field we obtain the following result:

Corollary 1. Let ν be a Gibbs state [given by (19) and (20)] on the configuration space $\Omega_\infty = S^{L_\infty(q)}$. Let $N = 0$ and $K \geqslant 0$. Define

$$\alpha = \inf\{[r, r]: r \in S\}, \qquad \beta = \sup\{[r, s]: r \neq s \text{ in } S\} \tag{33}$$

Then, if one of the following conditions is satisfied,

$$\begin{cases} K(\alpha - \beta) > \log(|S| - 1) \\ K(\alpha - \beta) \geqslant \log(|S| - 1) \text{ and } \exists! r_0 \in S \text{ such that } \alpha = [r_0, r_0] \end{cases} \tag{34}$$

then the probability vector $\pi = (\nu\{\sigma \in \Omega_\infty : \sigma(0) = r\}: r \in S)$ satisfies the equation

$$\pi = (h \circ A \circ \varphi)\pi, \qquad \pi \in P_1 \tag{35}$$

Proof. If $K(\alpha - \beta) - \log(|S| - 1) > 0$, note that

$$\delta = (\exp K\alpha) - (|S| - 1) \exp K\beta > 0$$

Then

$$\begin{aligned} \gamma = \sum_{r,s} a(r, s) z_r z_s &\geqslant \sum_{r \in S} a(r, r) z_r^2 - \sum_{r \neq s} a(r, s)\, |z_r|\, |z_s| \\ &\geqslant \delta \sum_r z_r^2 + (\exp K\beta)(|S| - 1) \sum_r z_r^2 - \sum_{r \neq s} |z_r|\, |z_s| \\ &\geqslant \delta \sum_r z_r^2 + \tfrac{1}{2}(\exp K\beta) \sum_{r \neq s} (|z_r| - |z_s|)^2 \end{aligned}$$

Then $\gamma \geqslant 0$ and $\gamma = 0$ iff $z = 0$. So A is positive definite and the result follows from Theorem 3(a). If the second condition of (34) is satisfied, we put

$$\delta' = \inf\{\exp K[r, r] - \exp K\alpha_0 : r \neq r_0\}$$

which is >0. Then

$$\gamma \geqslant \delta' \sum_{r \in S \setminus \{r_0\}} z_r^2 + \tfrac{1}{2}(\exp K\beta) \sum_{r \neq s} (|z_r| - |z_s|)^2 \geqslant 0$$

Let $\pi' = (h \circ A \circ \varphi)\pi$ and $z = \varphi(\pi') - \varphi(\pi')$. By using the proof of Theorem 3, we deduce $\gamma = \sum_{r,s} a(r, s)\, z_r z_s \leqslant 0$. Then $\gamma = 0$, but this occurs iff $z_r = 0$, $\forall r \in S\setminus\{r_0\}$. Since $\sum_{r \in S} \pi_r = \sum_{r \in S} \pi'_r$, we deduce $\pi = \pi'$. ∎

Examples

(a) If $S = \{-1, 1\}$ or $S = \{0, 1\}$, we deduce that the fixed-point equation (35) is satisfied by π for any positive ferromagnetic interaction $K > 0$ (and $N = 0$). In fact, if $S = \{-1, 1\}$, then $\alpha = 1$, $\beta = -1$, and the first condition in (34) is satisfied. When $S = \{0, 1\}$ we have that the second condition in (34) is satisfied by taking $r_0 = 0$. The solution in this case will be given in Section 5.

(b) Suppose that all the spin vectors are unitary, $[r, r] = 1$, $\forall r \in S$. Then

$$\beta = \inf\{\cos \theta(r, s) : r \neq s\}$$

where $\theta(r, s)$ is the angle between spins r, s. The first condition in (34) reads $K > (\log |S| - 1)(1 - \beta)^{-1}$. Then if $|S| = 2$ and the spin vectors are unitary and different, the fixed-point condition (35) is satisfied for any $K > 0$ (and $N = 0$).

4. PERIOD $\leqslant 2$ LIMIT ORBITS OF SOME NONLINEAR DYNAMICS ON $\mathbb{R}_+^S$

Some other results concerning the 2-period limit cycles for nonlinear dynamics on $\mathbb{R}_+^S$ can be established following the proof of our last result.

The properties of A that we will require are the following:

$$\begin{cases} A \text{ is symmetric for inner product } \langle\,,\,\rangle \text{ in } \mathbb{R}^S: \\ \langle Ax, y\rangle = \langle y, Ax\rangle \quad \forall x, y \in \mathbb{R}^S \end{cases} \tag{36}$$

$$A(\mathbb{R}_+^S \setminus \{0\}) \subset \mathbb{R}_+^S \setminus \{0\} \tag{37}$$

Recall that A given by (23) satisfies these properties. In fact, it satisfies $a(r, s) = a(s, r)$, $\forall s, r$, which is equivalent to satisfying (36) for the usual inner product in $\mathbb{R}^S$ and it is strictly positive, $A > 0[a(r, s) > 0, \forall r, s \in S]$, which is a sufficient condition in order that (37) is satisfied.

Other sufficient conditions weaker than $A > 0$ for (37) are deduced in the following results (which include conditions usually improved by stochastic matrices). Recall that the period of a state $r \in S$ with respect to a matrix $A \geqslant 0$ is $\eta_r = \text{u.c.d.}\{n: a^{(n)}(r, r) > 0\}$ and if A is irreducible, all the periods $\{\eta_r: r \in S\}$ are equal; we call this common value the period of A.

Lemma 1. Let $A \geqslant 0$ be irreducible and symmetric with respect to the inner product $\langle x, y \rangle_\varepsilon = \sum_{r \in S} \varepsilon_r x_r y_r$, where $\varepsilon_r > 0$, $\forall r \in S$. Then $A(\mathbb{R}^S_+ \backslash \{0\}) \subset \mathbb{R}^S_+ \backslash \{0\}$. Also in this case the matrix A is of period 1 or 2.

Proof. Let $x \in \mathbb{R}^S_+ \backslash \{0\}$, so $\exists r \in S$ such that $x_r \neq 0$. If $Ax = 0$, then $a(s, r) = 0$, $\forall s \in S$. Now, A being symmetric with respect to $\langle \, , \rangle_\varepsilon$ is equivalent to $\varepsilon_r a(r, s) = \varepsilon_s a(s, r)$; then we deduce $a(r, s) = 0$, $\forall r \in S$. The row and column r are 0, which contradicts the irreducibility of A.

To prove that A is of period 1 or 2, it suffices to show $a(r, r)^{(2n)} \neq 0$, $\forall n > 0$, $\forall r \in S$, where $a(r, r)^{(2n)}$ is the (r, r) term of A^n. By putting $a(s, s)^{(0)} = 1$, $\forall s \in S$, we have

$$a(r, r)^{(2n)} \geqslant \sum_{s \in S} a(r, s) a(s, s)^{(2n-2)} a(s, r) \qquad \text{for} \quad n \geqslant 1$$

Let n be the small integer such that for some $r \in S$, $a(r, r)^{(2n)} = 0$. Since $a(s, s)^{(2n-2)} > 0$, $\forall s \in S$, we get $a(r, s) a(s, r) = 0$, $\forall s \in S$; then by symmetry of A with respect to $\langle \, , \rangle_\varepsilon$ we deduce $a(r, s) = 0 = a(s, r)$, $\forall s \in S$, which contradicts the irreducibility of A. ∎

Theorem 6. Let A be symmetric, $A \geqslant 0$, such that $A(\mathbb{R}^S_+ \backslash \{0\}) \subset \mathbb{R}^S_+ \backslash \{0\}$ (for instance, if it satisfies the hypotheses of Lemma 1). Take

$$T(x) = \frac{Ax^p}{\|Ax^p\|} \qquad \text{for} \quad x \in \mathbb{R}^S_+ \backslash \{0\}, \quad p > 1, \quad T(0) = 0 \tag{38}$$

for some norm $\|\cdot\|$. Let $\Theta' \subset \mathbb{R}^S_+$ be the limit set of T. Then the restriction $T: \Theta' \to \Theta'$ is homeomorphism such that $T^2_{\Theta'} = id_{\Theta'}$, the identity on Θ'.

Proof. We can decompose $T = h \circ A \circ \varphi$, where $\varphi(x) = x^p$, $h(x) = x/\|x\|$ if $x \neq 0$, $h(0) = 0$. By the hypotheses, T maps $\mathbb{R}^S_+ \backslash \{0\}$ into $P_{\|\cdot\|} = \{x \in \mathbb{R}^S_+ : \|x\| = 1\}$. Then the limit set of T is $\Theta' = \{0\} \cup \Theta$, where $\Theta \subset P_{\|\cdot\|}$ is the limit set of the restriction of T to $\mathbb{R}^S_+ \backslash \{0\}$.

If $\|x\| = \|x\|_{p+1}$, then $f(x) = \varphi \circ h(x) = (x/\|x\|_{p+1})^p$ if $x \neq 0$, $f(0) = 0$;

it belongs to the subdifferential of $\|x\|_{p+1}$ and is equal to $f(x)$ of (30) with $p=q-1$. Now the rest of the proof is analogous to that of Theorem 5 because it only depends on f (the other facts used depend on the continuity of T on $P_{\|\cdot\|_{p+1}}$).

Now we shall prove the result for any norm $\|\cdot\|$. We denote by $h_p, T_p, D_p, \Theta'_p=\{0\}\cup\Theta_p$ the functions and sets related to $\|x\|_{p+1}$. We have $h_p\circ h=h_p$, $h\circ h_p=h$, $T\circ h_p=T$, $T_p\circ h=T_p$, $h_p\circ T=h_p\circ T_p$, $h\circ T_p=h\circ T$ on $\mathbb{R}_+^S$, so induction gives $h_p\circ T^n=T_p^n$, $h\circ T_p^n=T^n$ on $\mathbb{R}_+^S$. Let $y\in\Theta$; then $h_p(y)\in\Theta_p$, so $T_p^2\circ h_p(y)=h_p(y)$. This implies $h\circ T_p^2\circ h_p(y)=y$; then $T^2\circ h_p(y)=y$; finally, we deduce $T^2y=y$. Obviously T is continuous on the compact Θ', as $T^2_{\Theta'}=id_{\Theta'}$, it is a homeomorphism. ∎

Let us summarize the results obtained for the nonlinear evolution of probability vectors.

Corollary 2. Let $P_1=\{x\in\mathbb{R}_+^S: \sum_{r\in S}x_r=1\}$, and A be symmetric, $A\geqslant 0$, such that $A(\mathbb{R}_+^S\backslash\{0\})\subset\mathbb{R}_+^S\backslash\{0\}$. Let $p>0$. Then the following nonlinear evolutions on P_1,

$$(Tx(n))_r=x_r(n+1)$$
$$=\sum_{s\in S}a(r,s)x_s(n)^p\Big/\sum_{r'\in S}\sum_{s\in S}a(r',s)x_s(n)^p \tag{39}$$

$$(Tx(n))_r=x_r(n+1)$$
$$=\left(\sum_{s\in S}a(r,s)x_s(n)^{p/(p+1)}\right.$$
$$\left.\times\left\{\sum_{r'\in S}\left[\sum_{s\in S}a(r',s)x_s(n)^{p/(p+1)}\right]^{p+1}\right\}^{-p/(p+1)}\right)^{p+1} \tag{40}$$

have only limit probability vectors whose orbits are of period 1 or 2.

Proof. The result for the transformation (39) is just Theorem 6 for $\|x\|=\|x\|_1=\sum_{r\in S}x_r(x\in\mathbb{R}_+^S)$ and (40) is the evolution studied in Theorem 5. ∎

Now, if we see (39) as a generalization of a symmetric Markov chain, it is more convenient to write it as follows:

Corollary 3. Let A be irreducible, $A\geqslant 0$, A symmetric with respect to $\langle\cdot\rangle_\varepsilon$ for some $\varepsilon=(\varepsilon_r>0: r\in S)$. Let $p>0$. Then any limit probability vector of T given by (39) has period 1 or 2.

Proof. It follows by Lemma 1 that $A(\mathbb{R}_+^S\backslash\{0\})\subset\mathbb{R}_+^S\backslash\{0\}$; then we apply Corollary 2 to obtain the result. ∎

Now we shall study other nonlinear evolutions on $\mathbb{R}_+^S$ that also give rise to period $\leqslant 2$ limit points. We study functions that generalize $\varphi(x) = x^p$ and we normalize their actions by means of the Orlicz norm that they induce.

Let $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing, right continuous function such that $\phi(0) = 0$, $\phi(u) > 0$ when $u > 0$ and $\phi(\infty) = \lim_{u \to \infty} \phi(u) = \infty$.

We say that ϕ satisfies the (H) conditions.

Also consider $G_\phi(u) = \int_0^u \phi(u')\, du'$.

Let S be finite; G_ϕ induces a norm $\mathbb{R}^S$ called the Orlicz norm (see ref. 4 for a proof it is a norm):

$$\|x\|_\phi = \inf \left\{ k > 0 \colon \sum_{r \in S} G_\phi\left(\frac{|x_r|}{k}\right) \leqslant 1 \right\}$$

For $x \neq 0$ its norm $\|x\|_\phi$ is the unique $k > 0$ satisfying the equality

$$\sum_{r \in S} G_\phi\left(\frac{|x_r|}{k}\right) = 1, \qquad k = \|x\|_\phi \tag{41}$$

Recall that the class of functions $\phi_p(u) = [1/(p+1)]\, u^p$, $p > 0$, satisfy the above conditions and $\|x\|_{\phi_p} = \|x\|_{p+1}$.

On $\mathbb{R}_+^S$ we take

$$\varphi_\phi(x) = (\phi(x_r) \colon r \in S), \qquad h_\phi(x) = \frac{x}{\|x\|_\phi} \quad \text{if} \quad x \neq 0, \quad h_\phi(0) = 0 \tag{42}$$

Theorem 7. Let A be irreducible, $A \geqslant 0$, and symmetric with respect to some $\langle \cdot \rangle_\varepsilon$; ϕ satisfies the (H) conditions and also is continuous, strictly increasing. Then

$$T_\phi = h_\phi \circ A \circ \varphi_\phi \tag{43}$$

is a homeomorphism when restricted to its limit set Θ' and satisfies $((T_\phi)_{\Theta'})^2 = id_{\Theta'}$ (the identity on Θ').

Proof. An easy computation on (41) shows that $\|\cdot\|_\phi$ is differentiable on $\mathbb{R}_+^S \backslash \{0\}$ and its derivate

$$\hat{\varphi}_\phi(x) = \left(\frac{\partial}{\partial x_r} \|x\|_\phi \colon r \in S \right)$$

is given by

$$\begin{aligned} \hat{\varphi}_\phi(x) &= \gamma\left(\frac{x}{\|x\|_\phi}\right) \varphi_\phi\left(\frac{x}{\|x\|_\phi}\right) \\ \gamma(y) &= \left[\sum_{r \in S} y_r \phi(y_r) \right]^{-1} > 0 \end{aligned} \tag{44}$$

Define $\hat{\varphi}_\phi(0)=0$; then the equality $h_\phi \circ A \circ \hat{\varphi}_\phi = h_\phi \circ A \circ \varphi_\phi$ on $\mathbb{R}_+^S$. Hence we shall use the decomposition $T_\phi = h_\phi \circ A \circ \hat{\varphi}_\phi$. As $f=\hat{\varphi}_\phi \circ h_\phi = \hat{\varphi}_\phi$ on $\mathbb{R}_+^S$, the norm $\|x\|_\phi$ is a convex potential associated to f. Then $\|A\hat{\varphi}_\phi(T_\phi^n x)\|_\phi$ increases with $n \geqslant 1$ for any $x \in \mathbb{R}_+^S$ (this follows from Theorem 4).

The limit set of T_ϕ is $\Theta' = \{0\} \cup \Theta$, with $\Theta \subset P_\phi = \{x \in \mathbb{R}_+^S : \|x\|_\phi = 1\}$ and $T_\phi(0)=0$. Let $y \in \Theta$; as T_ϕ is continuous on the compact P_ϕ, there exists $z \in \Theta$ such that $y = T_\phi(z)$. By taking $v(n) = A\hat{\varphi}_\phi(T_\phi^n z)$, we deduce, as in the proof of Theorem 5, that $\|v(n)\|_\phi = M$ a constant >0 for any $n \geqslant 0$, and also $\langle \hat{\varphi}_\phi(v(2)), v(2)-v(0) \rangle = 0$.

Let $w(n) = v(n)/\|v(n)\|_\phi = v(n)/M$. By definition of $\|\cdot\|_\phi$ we have $\sum_{r \in S} G_\phi(w_r(n)) = 1$. As

$$\hat{\varphi}_\phi(v(n)) = \hat{\varphi}_\phi(Mw(n)) = \gamma(w(n))\, \varphi_\phi(w(n))$$

where $\gamma(w(n)) \neq 0$, we deduce $\langle \varphi_\phi(w(2)), w(2)-w(0) \rangle = 0$.

Now define $\tilde{G}_\phi(x) = \sum_{r \in S} G_\phi(x_r)$, which is a potential of φ_ϕ strictly convex because ϕ is strictly increasing. From the above equalities we deduce

$$\tilde{G}_\phi(w(2)) - \tilde{G}_\phi(w(0)) = \langle \varphi(w(2)), w(2)-w(0) \rangle = 0$$

From strict convexity of $\tilde{G}_\phi$, $w(2)=w(0)$, which implies $T_\phi^2 y = y$. Then we obtain the result. ∎

A version of the above result as a nonlinear dynamics on probability vectors can be established. Let $G_\phi(x) = (G_\phi(x_r): r \in S)$ and G_ϕ^{-1} be its inverse in $\mathbb{R}_+^s$.

Corollary 4. Let A, ϕ satisfy the hypotheses of Theorem 7. Then the limit probability vectors of the evolution

$$\pi(n+1) = G_\phi^{-1} \circ h_\phi \circ A \circ \varphi_\phi \circ G_\phi^{-1} \pi(n) \tag{45}$$

are of period 1 or 2.

Proof. Note that

$$T' = G_\phi \circ h_\phi \circ A \circ \varphi_\phi \circ G_\phi^{-1} = G_\phi \circ h_\phi \circ A \circ \hat{\varphi}_\phi \circ G_\phi^{-1}$$

maps P_1 into itself; in fact, $h_\phi \circ A \circ \hat{\varphi}_\phi \circ G_\phi^{-1}$ maps P_1 into P_ϕ, and G_ϕ maps P_ϕ into P_1. Now take $h' = G_\phi \circ h_\phi$, $\varphi' = \hat{\varphi}_\phi \circ G_\phi^{-1}$; then $\varphi' \circ h' = \hat{\varphi}_\phi \circ h_\phi$, and by the proof of the last theorem we deduce the result. ∎

5. FINAL COMMENTS

A. Theorem 6 also gives us information about the dynamics of the transformation $(\tau_{(p)}(x))_r = \sum_{s \in S} a(r, s)\, x_s^p$ on $\mathbb{R}_+^S$. In fact, $\tau_{(p)}(\lambda x) = \lambda^p \tau(x)$ for $\lambda \geqslant 0$; then the images of two points in the same ray also belong to the same ray. Hence $\tau_{(p)}$ induces a transformation among the rays of $\mathbb{R}_+^S$ that is just given by $T(x)$ of (38) (we can use any norm $\|\cdot\|$). Then $\tau_{(p)}$ accummulates only orbits (of rays) of period $\leqslant 2$ in this action.

B. To get the one-site distribution of the Gibbs states in the Bethe lattice we must solve Eq. (22), $\pi = T^2\pi$, $\pi \in P_1$, where $T = h \circ A \circ \varphi$ is given by (23) and (24). From Theorems 5 and 6 and the Remark after Theorem 5 we can first solve the equation

$$\begin{aligned} &\xi = (h' \circ A \circ \varphi')^2\, \xi, \qquad && \xi \neq 0 \\ &\varphi'(x) = x^{q-1}, && h'(x) = x \Big/ \sum_{r \in S} x_r \end{aligned} \tag{46}$$

Then the probability vector π is given by

$$\pi = (\xi / \|\xi\|_q)^q \tag{47}$$

Let us note that $p = q - 1$, which is an integer > 0, and suppose that the number of spins is $|S| = 2$. Then ξ satisfying (46) is a couple $\xi = (m, 1 - m)$ satisfying

$$A(A\xi^p)^p = \lambda\xi \qquad \text{for some} \quad \lambda > 0 \tag{48}$$

Note that $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$; then the equation for ξ is reduced to

$$\begin{aligned} &m\{b[am^p + b(1-m)^p]^p + c[bm^p + c(1-m)^p]^p\} \\ &\quad (1-m)\{a[am^p + b(1-m)^p]^p + b[bm^p + c(1-m)^p]^p\} = 0 \\ &\quad m \in [0, 1] \end{aligned} \tag{49}$$

Now the biggest exponent in the above polynomial in m is m^{p^2+1}; its coefficient is

$$\sum_{k=0}^{p} \binom{p}{k} (a^{k+1}b^{p-k} + b^{k+1}c^{p-k} + a^k b^{p+1-k} + b^k c^{p+1-k})(-1)^{p(p-k)} \tag{50}$$

Then, if q is odd and $p = 2p'$ is even, $(q-1)^2 + 1$ is a bound for the different values of π satisfying Eq. (22). For q even it can happen that all the coefficients of the polynomial (49) vanish; in this case any probability vector π satisfies (22). If the polynomial (49) is not identically zero, its degree $\leqslant (q-1)^2+1$ is a bound for the number of solutions π. If $|S|>2$, the analysis of solutions becomes complicated.

C. Consider the case $q=2$ (so $p=1$) and S any finite spin set. Suppose $A \geqslant 0$, A^2 irreducible (for instance $A>0$), Perron–Frobenius theory implies equation (48): $A^2\xi=\lambda\xi$ for $\xi \geqslant 0$, $\xi \neq 0$; has only one solution. Such ξ also verifies $A\xi=\sqrt{\lambda}\,\xi$, then it is a fixed point for transformation T.

Let us study the simplest case $q=2$, $|S|=2$. Note as before $A=\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$. If $a=c=0$ [which does not happen for the matrix A given by (23) because $A>0$], then $A=b\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $b \neq 0$. Then $A^2\xi=\lambda\xi$ is satisfied for any $\xi \in \mathbb{R}_+^2$ with $\lambda=b^2$; then any vector π in P_1 is a solution. $\pi=(1/2, 1/2)$ is the only fixed-point solution, that is, that verifies $\pi=T\pi$; any other solution is such that $\pi \neq T\pi$.

If $a+c>0$, Eq. (49) becomes

$$(a-c)\,m^2-[(a-c)-2b]\,m-b=0 \tag{51}$$

which is the same equation we get if we impose $A\xi=\lambda\xi$ for some $\lambda>0$. Then any solution π is in this case a fixed-point solution $\pi=T\pi$.

If $a=c$, there is only one solution for (51): $m=1-m=1/2$; then $\pi=(1/2, 1/2)$. Now $a=c$ is satisfied when $S=\{-1, 1\}$, $N=0$. Let $a \neq c$. Let $\delta=b/a-c$. If $\delta=0$, we get two solutions, $m=0$, $m=1$. For $\delta>0$ it is easily shown that the unique solution in $[0, 1]$ is $m(\delta)=1/2-\delta+1/2(1+4\delta^2)^{1/2}$. For $\delta<0$ the unique solution is $1-m(-\delta)$.

D. If A is not symmetric, our results cannot be obtained. In fact, if

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \cdots & \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is the cyclic matrix of order n, then we have finite orbits of order n under the evolution $T=h \circ A \circ \phi$ for any ϕ, h constructed in Sections 3 and 4. We can also see that orbits of length n continue to exist under small perturbations, that is, for

$$A=\begin{pmatrix} \varepsilon & 1 & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \varepsilon & 1 & \cdots & \varepsilon \\ \vdots & & & & \cdots \\ \varepsilon & \varepsilon & \varepsilon & \cdots & 1 \\ 1 & \varepsilon & \varepsilon & \cdots & \varepsilon \end{pmatrix}$$

where $0<\varepsilon \ll 1$ (then $A>0$).

E. When $0 \in A(\mathbb{R}_+ \setminus \{0\})$, the results obtained in Section 4 for any one of the transformations $T = h \circ A \circ \varphi$ studied are little modified. In fact, it can be shown that the orbits of the limit points are of the form $(y, z, y, z,...)$ or $(y, z, 0, 0,...)$ with $y, z \in \mathbb{R}_+^S$.

F. If A is symmetric positive definite [for instance, if A satisfies the first condition of (34)], the solutions satisfy the following "generalized eigenvalue equation":

$$A\xi^p = \lambda\xi \qquad \text{with} \quad \lambda > 0, \quad \xi \geqslant 0, \quad \sum_{r \in S} \xi_r = 1 \tag{52}$$

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